§3.5 Diagramatic Proof of Gauge Invariance

Recall:

The Lagrangian

$$\mathcal{L} = \bar{\psi}\left[i\gamma^\mu(\partial_\mu - ieA_\mu) - m\right]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (1)$$

is "gauge invariant" in the sense that under

$$\psi(x) \longmapsto e^{ie\varepsilon(x)}\psi(x) \qquad (2a)$$

$$A_\mu(x) \longmapsto A_\mu(x) + \partial_\mu \varepsilon(x) \qquad (2b)$$

$\mathcal{L}$ stays invariant (note $F_{\mu\nu} \mapsto F_{\mu\nu}$)

trf. (2b) can alternatively be written as

$$A_\mu \longmapsto A_\mu(x) - i e^{-i\varepsilon(x)} \partial_\mu e^{i\varepsilon(x)} \qquad (3)$$

From the form (3) one can see that $\varepsilon(x)$ is equivalent to $\varepsilon + 2\pi$

We will come back to this later

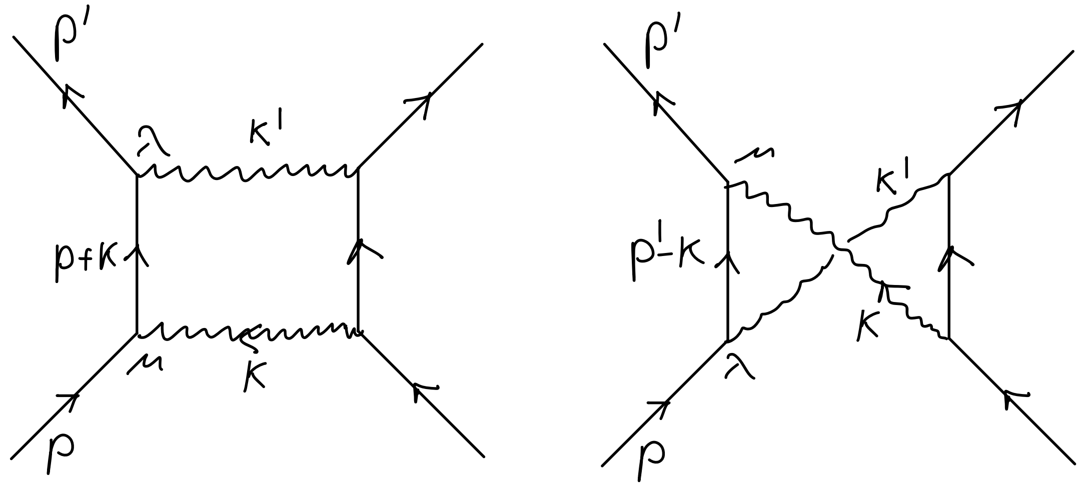
In order to write down the photon propagator, we add to (1) a mass term $\mu^2 A_\mu A^\mu$ and read off

$$i D_{\mu\nu} = \frac{i(k_\mu k_\nu/\mu^2 - \eta_{\mu\nu})}{k^2 - \mu^2}$$

μ serves here as a "regulator" and we want to show that in Feynman diagrams, we can safely take the limit $\mu \to 0$

<u>A specific example</u>:

Consider electron-electron scattering to order e^4:



$\longrightarrow$ the amplitude is:

$$\bar{u}(p')\left(\gamma^\lambda \frac{1}{\not p + \not k - m}\gamma^\mu - \gamma^\mu \frac{1}{\not p' - \not k - m}\gamma^\lambda\right) u(p)$$

$$\times \frac{i}{k^2}\left(\frac{k_\mu k_\nu}{\mu^2} - \eta_{\mu\nu}\right)\Gamma_\lambda{}^\nu \qquad (4)$$

where the specific form of $\Gamma_\lambda{}^\nu$ does not concern us

$\longrightarrow$ contracting the terms with k_μ gives

$$\bar{u}(p')\left(\gamma^\lambda \frac{1}{\not p + \not k - m}\not k + \not k \frac{1}{\not p' - \not k - m}\gamma^\lambda\right) u(p)$$

$$= \bar{u}(p')\left(\gamma^\lambda \frac{(\not p + \not k - m) - (\not p - m)}{\not p + \not k - m} + \frac{(\not p' - m) - (\not p' - \not k - m)}{\not p' - \not k - m}\gamma^\lambda\right) u(p)$$

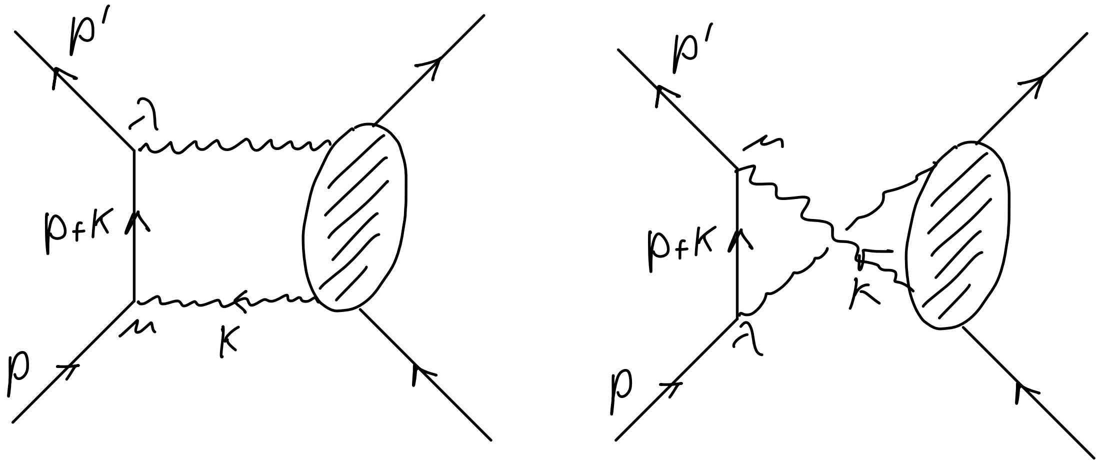
$$(\not p - m)\, u(p) = 0\,, \quad \bar{u}(p')(\not p' - m) = 0$$

$$= \bar{u}(p')\left(\underbrace{\gamma^\lambda \frac{\not p + \not k - m}{\not p + \not k - m} - \frac{\not p' - \not k - m}{\not p' - \not k - m}\gamma^\lambda}_{=0}\right) u(p)$$

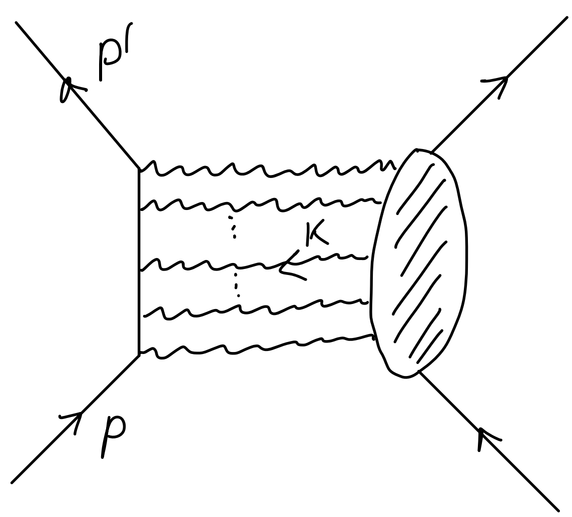
$= 0$

$\longrightarrow$ can safely take the limit $\mu^2 \to 0$ in this case!

Since the explicit form of $T_\lambda{}^\nu$ did not enter the calculation, we could have replaced our diagram by the more general:



We can even generalize to



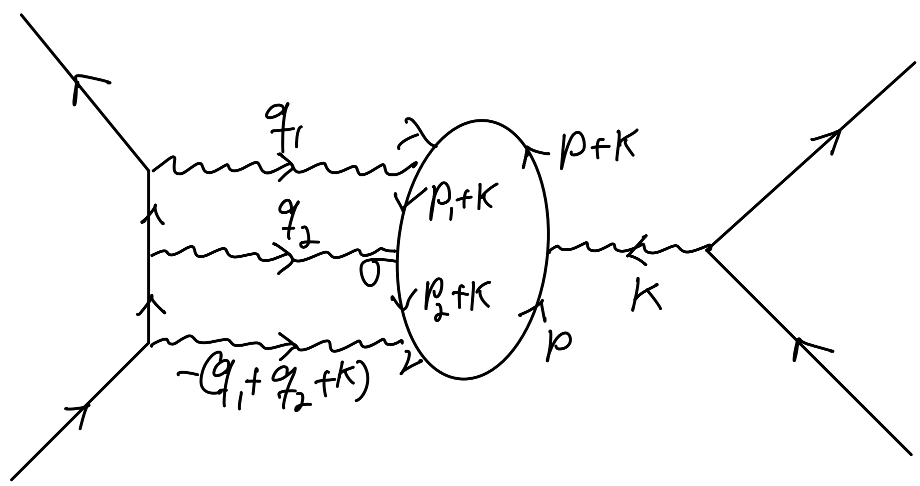
(exercise)

Photon landing on an internal line

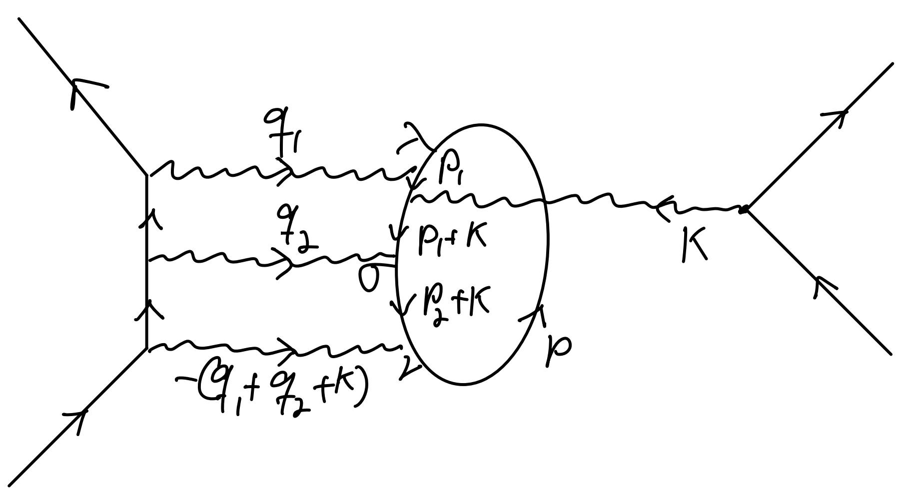
In above examples, photon landed on "external line" $\rightarrow$ used e.o.m. for $\bar{u}(p')$ and $u(p)$

What if photon ends on "internal line"?

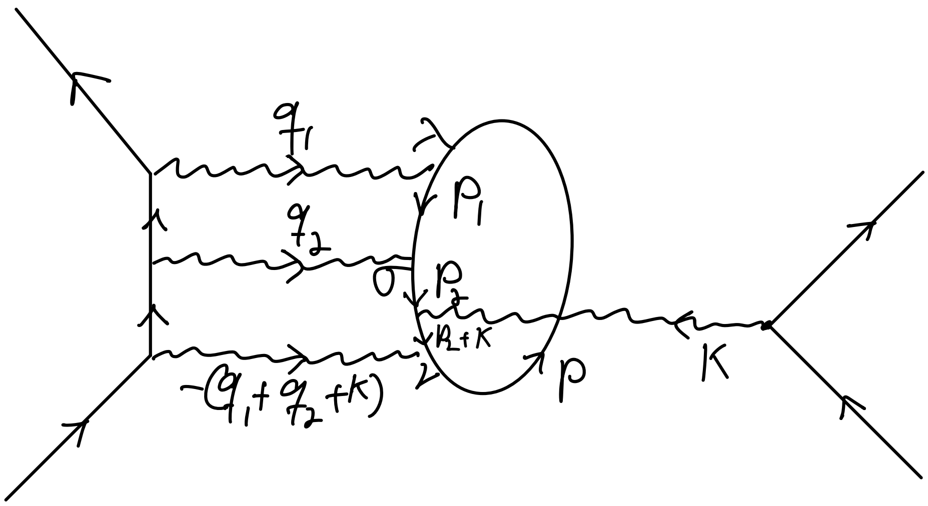
Consider electron-electron scattering to order e^8 $(p_1 = p + q_1,\ p_2 = p_1 + q_2)$:



(a)



(b)



(c)

Want to show that $K_\mu K_\rho/\mu^2$ in photon propagator does not contribute

$\longrightarrow$ to save writing, replace it by K_μ

then :

$$(a) = \int \frac{d^4p}{(2\pi)^4}\, tr\left(\gamma^\nu \frac{1}{\not p_2 + \not K - m}\gamma^\sigma \frac{1}{\not p_1 + \not K - m}\gamma^\lambda \times \frac{1}{\not p + \not K - m}\not K \frac{1}{\not p - m}\right)$$

$$(b) = \int \frac{d^4p}{(2\pi)^4}\, tr\left(\gamma^\nu \frac{1}{\not p_2 + \not K - m}\gamma^\sigma \frac{1}{\not p_1 + \not K - m}\not K \times \frac{1}{\not p_1 - m}\gamma^\lambda \frac{1}{\not p - m}\right)$$

$$(c) = \int \frac{d^4p}{(2\pi)^4}\, tr\left(\gamma^\nu \frac{1}{\not p_2 + \not K - m}\not K \frac{1}{\not p_2 - m}\gamma^\sigma \frac{1}{\not p_1 - m}\gamma^\lambda \frac{1}{\not p - m}\right)$$

Now substitute in (c) $\not{k} = (\not{p}_2 + \not{k} - m) - (\not{p}_2 - m)$

$$\longrightarrow (c) = \int \frac{d^4p}{(2\pi)^4} \left[tr\left(\gamma^\nu \frac{1}{\not{p}_2 - m} \gamma^\sigma \frac{1}{\not{p}_1 - m} \gamma^\lambda \frac{1}{\not{p} - m}\right) \right.$$

$$\left. - tr\left(\gamma^\nu \frac{1}{\not{p}_2 + \not{k} - m} \gamma^\sigma \frac{1}{\not{p}_1 - m} \gamma^\lambda \frac{1}{\not{p} - m}\right)\right]$$

In (b) substitute $\not{k} = (\not{p}_1 + \not{k} - m) - (\not{p}_1 - m)$

$$\longrightarrow (b) = \int \frac{d^4p}{(2\pi)^4} \left[tr\left(\gamma^\nu \frac{1}{\not{p}_2 + \not{k} - m} \gamma^\sigma \frac{1}{\not{p}_1 - m} \gamma^\lambda \frac{1}{\not{p} - m}\right) \right.$$

$$\left. - tr\left(\gamma^\nu \frac{1}{\not{p}_2 + \not{k} - m} \gamma^\sigma \frac{1}{\not{p}_1 + \not{k} - m} \gamma^\lambda \frac{1}{\not{p} - m}\right)\right]$$

Finally, in (a) write $\not{k} = (\not{p} + \not{k} - m) - (\not{p} - m)$

$$\longrightarrow (a) = \int \frac{d^4p}{(2\pi)^4} \left[tr\left(\gamma^\nu \frac{1}{\not{p}_2 + \not{k} - m} \gamma^\sigma \frac{1}{\not{p}_1 + \not{k} - m} \gamma^\lambda \frac{1}{\not{p} - m}\right) \right.$$

$$\left. - tr\left(\gamma^\nu \frac{1}{\not{p}_2 + \not{k} - m} \gamma^\sigma \frac{1}{\not{p}_1 + \not{k} - m} \gamma^\lambda \frac{1}{\not{p} + \not{k} - m}\right)\right]$$

Altogether:

$$(a) + (b) + (c) = \int \frac{d^4p}{(2\pi)^4} \left[tr\left(\gamma^\nu \frac{1}{\not{p}_2 - m} \gamma^\sigma \frac{1}{\not{p}_1 - m} \gamma^\lambda \frac{1}{\not{p} - m}\right) \right.$$

$$\left. - tr\left(\gamma^\nu \frac{1}{\not{p}_2 + \not{k} - m} \gamma^\sigma \frac{1}{\not{p}_1 + \not{k} - m} \gamma^\lambda \frac{1}{\not{p} + \not{k} - m}\right)\right]$$

shifting $p \mapsto p-k$ in the second term above, we see that the two terms cancel!

$\to$ $k_\mu k_\rho/\mu^2$ piece in the photon propagator goes away and we can set $\mu^2=0$

<u>Summary:</u>

Given any physical amplitude $T^{\mu\cdots}(k,\cdots)$ with external electrons on-shell, we have

$$k_\mu T^{\mu\cdots}(k,\cdots)=0$$

"Ward-Takahashi" identity

$\to$ we can write $iD_{\mu\nu} = \frac{-i\eta^{\mu\nu}}{k^2}$ for the photon propagator

More generally, we can use

$$iD_{\mu\nu} = \frac{i}{k^2}\left[(1-\xi)\frac{k_\mu k_\nu}{k^2} - \eta_{\mu\nu}\right]$$

can choose ξ freely

$\xi=1$: "Feynman gauge", $\xi=0$: "Landau gauge"